Shortcuts to Adiabaticity, Optimal Quantum Control, and Thermodynamics
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Transitionless quantum driving in open quantum systems

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OUTLINE

- Berry's transitionless quantum driving
- the rotating frame for unitary evolutions
- adiabatic theorem for open quantum systems
- transitionless quantum driving in open quantum systems
- examples

Berry's transitionless quantum driving

$$\hat{H}_0(t)|\varphi_n(t)\rangle = E_n(t)|\varphi_n(t)\rangle$$

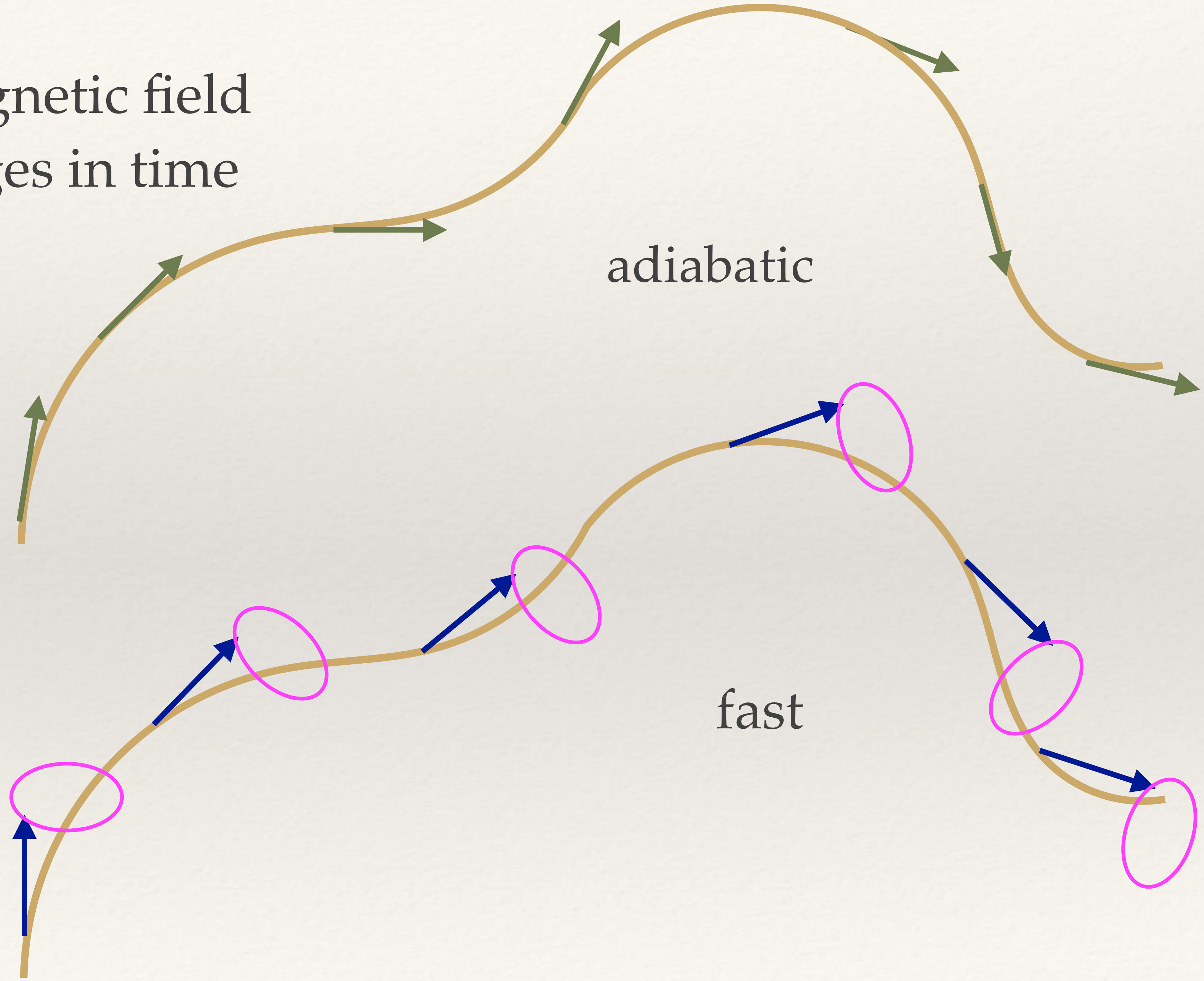
if the initial state of the system is an instantaneous eigenstate of a time dependent Hamiltonian Ho it will remain the the corresponding eigenstate at time t as long as Ho varies slowly enough and there are no level crossing

$$\hat{H}(t) = \hat{H}_0(t) + \hat{H}_1(t) \qquad \hat{H}(t)|\varphi_n(t)\rangle = i\partial_t|\varphi_n(t)\rangle$$

transitionless quantum driving: add a new time dependent hamiltonian term H_1 so that the state becomes an exact solution regardless of the speed of change of the hamiltonian

a classical analog

a spin following a magnetic field
whose direction changes in time



the rotating frame

$$\hat{H}(t) = \sum_{i,j} |i\rangle\langle i|\hat{H}(t)|j\rangle\langle j|$$

$$\hat{U}(t) = \sum_{i} |\varphi_i(t)\rangle\langle i|$$

U(t) diagonalizes the hamiltonian

$$\hat{U}^{-1}(t)\hat{H}(t)\hat{U}(t) \equiv \hat{H}_{\mathrm{d}}(t) = \sum_{i} E_i(t)|i\rangle\langle i|$$

the time dependent Schoedinger eq.

$$|\psi\rangle_{\mathrm{d}} = \hat{U}^{-1}|\psi\rangle.$$

time dependent unitary transformation

$$\hat{H}_{\mathrm{d}}(t) + i\partial_t \hat{U}^{-1}(t)\hat{U}(t)]|\psi\rangle_{\mathrm{d}} = i\partial_t|\psi\rangle_{\mathrm{d}}$$

$$[\hat{H}_{\mathrm{d}}(t) + \hat{H}'_{\mathrm{d}}(t) + \hat{H}'_{\mathrm{nd}}(t)]|\psi\rangle_{\mathrm{d}} = i\partial_t|\psi\rangle_{\mathrm{d}}.$$

$$\hat{H}'_{\mathrm{d}}(t) = i\sum |i\rangle\langle i|\partial_t \hat{U}^{-1}(t)\hat{U}(t)|i\rangle\langle i| = i\sum \langle\dot{\varphi}_i|\varphi_i\rangle|i\rangle\langle i|,$$

$$\hat{H}'_{\mathrm{nd}}(t) = i\sum_{i\neq j} |i\rangle\langle i|\partial_t \hat{U}^{-1}(t)\hat{U}(t)|j\rangle\langle j| = i\sum_{i\neq j} \langle\dot{\varphi}_i|\varphi_j\rangle|i\rangle\langle j|,$$

transitionless quantum driving

Berry connection

$$\hat{H}'_{\mathrm{d}}(t) = i\sum |i\rangle\langle i|\partial_t \hat{U}^{-1}(t)\hat{U}(t)|i\rangle\langle i| = i\sum \langle\dot{\varphi}_i|\varphi_i\rangle|i\rangle\langle i|,$$

non adiabatic term

$$\hat{H}'_{\mathrm{nd}}(t) = i\sum_{i\neq j} |i\rangle\langle i|\partial_t \hat{U}^{-1}(t)\hat{U}(t)|j\rangle\langle j| = i\sum_{i\neq j} \langle\dot{\varphi}_i|\varphi_j\rangle|i\rangle\langle j|,$$

transitional quantum driving

$$\hat{H}_{\mathrm{tqd}}(t) = -\hat{U}(t)\hat{H}'_{\mathrm{nd}}(t)\hat{U}^{-1}(t).$$

adiabatic approximation in open quantum systems

$$\mathcal{L}[\varrho] = -i[\hat{H}(t), \varrho] + \frac{1}{2}\sum_{j=1}^{N}(2\hat{\Gamma}_j(t)\varrho\hat{\Gamma}_j^\dagger(t) - \{\varrho, \hat{\Gamma}_j^\dagger(t)\hat{\Gamma}_j(t)\})$$

due to the coupling of the system with the environment, the energy-difference between neighbouring eigenvalues of the Hamiltonian no longer provides the natural time-scale with respect to which a time-dependent Hamiltonian could be considered to be slowly-varying.

adiabaticity of open systems is reached when the evolution of the state of a system occurs without mixing the various Jordan blocks into which L can be decomposed.

Sarandy Lidar PRA 71, 012331 (2005)

a matrix reppresentation of $\mathcal{L}$

define a time independent basis in the D2-dimensional space of the density matrices.This could consist, for example, the three Pauli matrices and the identity matrix in the case of a single spin-1/2.

$$B \equiv \{\hat{\sigma}_i\} \qquad i = \{1, \ldots, D^2\}.$$

the density operator becomes a vector $|\varrho\rangle\rangle = (\rho_1, \rho_2, \ldots, \rho_{D^2})^\dagger,$

the Lindblad operator becomes a super matrix $L(t)|\varrho\rangle\rangle = |\dot{\varrho}\rangle\rangle.$

$$\rho_j = \mathrm{Tr}[\hat{\sigma}_j^\dagger \varrho] \qquad L_{jk}(t) = \mathrm{Tr}[\hat{\sigma}_j^\dagger(\mathcal{L}_t[\hat{\sigma}_k])].$$

Jordan decomposition

Although the supermatrix L(t) might be non-Hermitian, in which case it cannot be diagonalized in general, it is always possible to find a similarity transformation C(t) such that L(t) is written in the canonical Jordan form

$$L_{\mathrm{J}}(t) = C^{-1}(t)L(t)C(t) = \mathrm{diag}[J_1(t), \ldots, J_N(t)],$$

$$C(t) = \sum_{\nu=1}^{N} \sum_{\mu_\nu=1}^{M_\nu} |\mathcal{D}_{\nu,\mu_\nu}(t)\rangle\rangle\langle\langle\sigma_{\nu,\mu_\nu}|,$$

$$L(t)|\mathcal{D}_{\nu,\mu_\nu}(t)\rangle\rangle = |\mathcal{D}_{\nu,\mu_\nu-1}(t)\rangle\rangle + \lambda_\nu(t)|\mathcal{D}_{\nu,\mu_\nu}(t)\rangle\rangle,$$

$|\mathcal{D}_{\nu,0}(t)\rangle\rangle$ represents the eigenvector of L(t) corresponding to the the eigenvalue $\lambda_\nu(t)$

transitionless open dynamics

formal analogy with the unitary case $(L_{\rm J} + L'_{\rm J} + L'_{\rm nd})|\varrho\rangle\rangle_{\rm J} = |\dot{\varrho}\rangle\rangle_{\rm J},$

$$L'_{\rm J} = \sum |\sigma_{\nu,\mu_\nu}\rangle\rangle\langle\langle\sigma_{\nu,\mu_\nu}|\dot{C}^{-1}C|\sigma_{\nu,\mu_\nu}\rangle\rangle\langle\langle\sigma_{\nu,\mu_\nu}|$$

$$L'_{\rm nd} = \sum_{\nu\neq\nu'} |\sigma_{\nu,\mu_\nu}\rangle\rangle\langle\langle\sigma_{\nu,\mu_\nu}|\dot{C}^{-1}C|\sigma_{\nu',\mu'_{\nu'}}\rangle\rangle\langle\langle\sigma_{\nu',\mu'_{\nu'}}|$$

transitionless quantum driving $L_{\rm tqd} = -CL'_{\rm nd}C^{-1}.$

the driving term can be unitary (hamiltonian)
or non unitary (a quantum channel)

for one dimensional Jordan blocks
the off diagonal
matrix term of the correction term are $\left\langle\left\langle \dot{\mathcal{D}}_i(t) \middle| \mathcal{D}_j(t)\right\rangle\right\rangle = \frac{\left\langle\left\langle \mathcal{D}_i(t) \middle| \dot{L}(t) \middle| \mathcal{D}_j(t)\right\rangle\right\rangle}{\lambda_j - \lambda_i}.$

rotating jump operators and unitary driving

$$\mathcal{L}[\rho] = \sum_k \frac{\gamma_k}{2}\left[2\hat{\Gamma}_k(t)\rho\hat{\Gamma}_k^{\dagger}(t) - \left\{\hat{\Gamma}_k^{\dagger}(t)\hat{\Gamma}_k(t), \rho\right\}\right],$$

$$\hat{\Gamma}_k(t) = \hat{U}^{\dagger}(t)\hat{\Gamma}_0^k\hat{U}(t)$$

in the rotating frame $\quad \tilde{\rho} = \hat{U}(t)\rho(t)\hat{U}^{\dagger}(t)$

$$\dot{\tilde{\rho}} = \sum_k \frac{\gamma}{2}\left[2\hat{\Gamma}_0^k\tilde{\rho}\hat{\Gamma}_0^k - \left\{\hat{\Gamma}_0^k\hat{\Gamma}_0^k, \tilde{\rho}\right\}\right] - i\left[i\dot{\hat{U}}(t)\hat{U}^{\dagger}(t), \tilde{\varrho}\right].$$

the quantum unitary driving $\quad \hat{H}_{\text{tqd}}(t) = i\dot{\hat{U}}(t)\hat{U}^{\dagger}(t).$

example 1: single spin amplitude damping

$$\mathcal{L}_{\mathrm{ad}}[\varrho] = \frac{\gamma}{2}[2\hat{\sigma}_{\boldsymbol{n}}^{-}\varrho\hat{\sigma}_{\boldsymbol{n}}^{+} - \{\hat{\sigma}_{\boldsymbol{n}}^{-}\hat{\sigma}_{\boldsymbol{n}}^{+}, \varrho\}]$$

$$\hat{\sigma}_{\boldsymbol{n}}^{-} = (\hat{\sigma}_{\boldsymbol{n}}^{+})^{\dagger} = |\downarrow\rangle_{\boldsymbol{n}}\langle\uparrow|$$

amplitude damping along a time dependent direction

$$n = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

precession around the z axis

$$\phi = \omega t$$

$$\hat{B} \equiv \left(\hat{\mathcal{I}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\right).$$

$$\mathcal{L}_{\mathrm{tqd}}[\varrho] = -i\left[\hat{H}_{\mathrm{tqd}}(t), \varrho\right]$$

$$\hat{H}_{\mathrm{tqd}}(t) = (n \times \dot{n}) \cdot \hat{\boldsymbol{\sigma}},$$

$$\hat{H}_{\mathrm{tqd}}(t) = i\dot{\hat{U}}\hat{U}^{\dagger},$$

a two qubit example

$$\Gamma_1 = \hat{U}\left(\left|0\right\rangle_1\left\langle 1\right| \otimes \hat{\mathcal{I}}_2\right)\hat{U}^\dagger; \quad \Gamma_2 = \hat{U}\left(\hat{\mathcal{I}}_1 \otimes \left|0\right\rangle_2\left\langle 1\right|\right)\hat{U}^\dagger$$

where U is a Hadamardt gate followed by a C-NOT

the liuvillian has the following fixed point: $|\psi\rangle = \left(1/\sqrt{2}\right)\left(|00\rangle + |11\rangle\right)$

let's generalise by assuming U is an arbitrary single qubit rotation followed by a C-NOT. In this case the fixed point is

$$\left|\psi\left(t\right)\right\rangle = \left(\cos\theta\left(t\right)|00\rangle + \sin\theta\left(t\right)|11\rangle\right)$$

by rotating q one can drag the fixed point

a two qubit example

by varying slowly θ one can drag the fixed point

such dragging can be achieved exactly with no constrains on speed by adding the following coherent driving:

$$H_{\mathrm{tqd}} = i\dot{\hat{U}}\hat{U}^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & -i\dot{\theta} \\ 0 & 0 & -i\dot{\theta} & 0 \\ 0 & i\dot{\theta} & 0 & 0 \\ i\dot{\theta} & 0 & 0 & 0 \end{pmatrix}$$

$$H_{\mathrm{tqd}} = -i\dot{\theta}\left(|00\rangle\langle 11| + |01\rangle\langle 10|\right) + \mathrm{h.c.}$$