# Quantum speedup by adiabatic state transformations and quantum annealing

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Joint work with

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Emanuel Knill
Daniel Nagaj
Guanglei Xu
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**References:**

S Boixo, E Knill, **RDS**, QIC 9, 0833 (2009)
S Boixo, E Knill, **RDS**, arXiv: 1005.3034 (2010)
**RDS**, S Boixo, arXiv: 1110.2494 (2011) - SICOMP
**RDS**, D Nagaj, M Kieferova, PRL 109, 050501 (2012)
Chiang, Xu, **RDS**, PRA 89, 012314 (2014)

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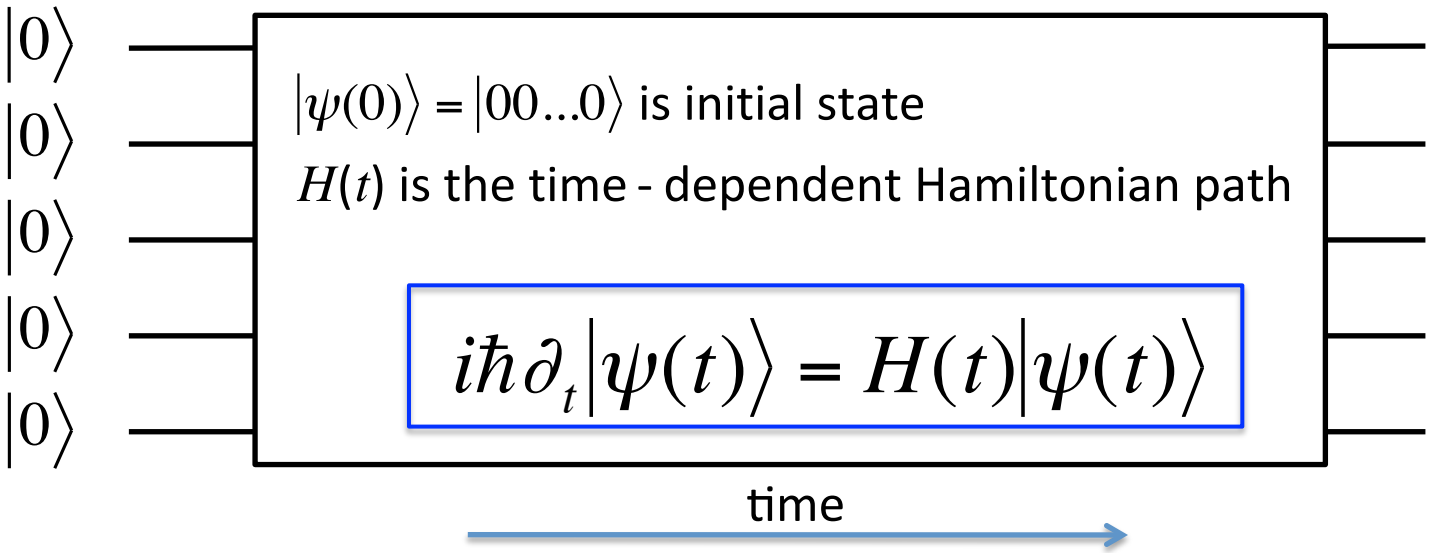
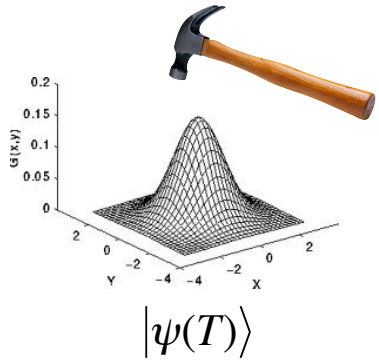
# Quantum Algorithms in Hamiltonian-based Models of QC




initial state : $|\psi(0)\rangle = |00\ldots0\rangle$

unitary evolution : $U = \tau\left\{\exp\left(-i\int_0^T H(t)dt\right)\right\}$

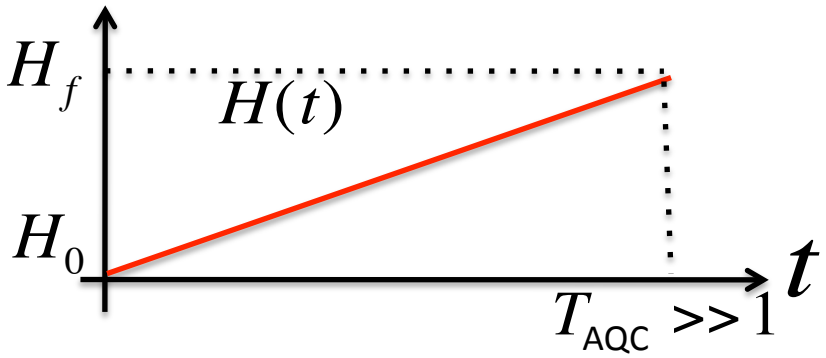
final state : $|\psi(T)\rangle = a_{00\ldots0}|00\ldots0\rangle + a_{10\ldots0}|10\ldots0\rangle + \ldots + a_{11\ldots1}|11\ldots1\rangle$

measurement : $|\sigma\rangle$ , $\Pr(\sigma) = |a_\sigma|^2$

Coherent evolution

# Adiabatic Quantum Computing

$$H_0 \xrightarrow{H(t)} H_f$$



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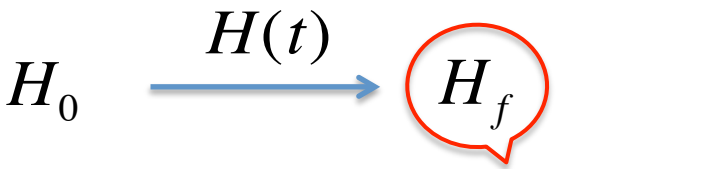
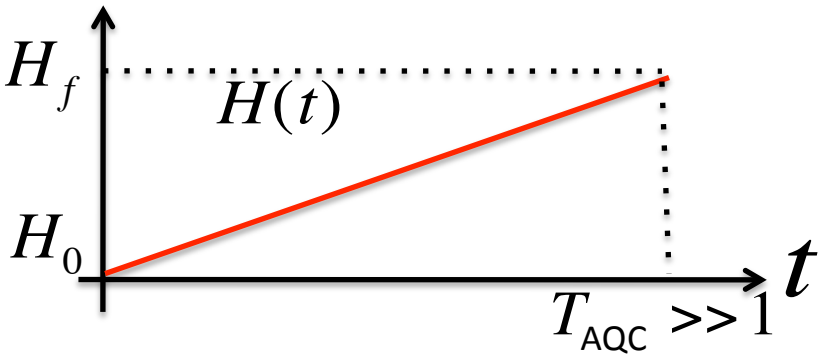



Input Hamiltonian



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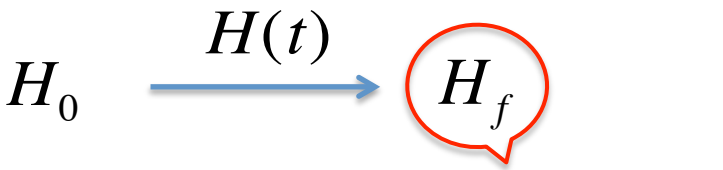


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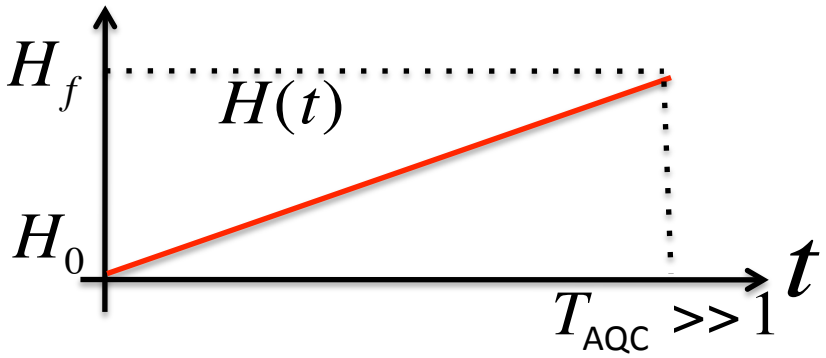


Eigenvalues : $\lambda_0(t) < \lambda_1(t) \leq \lambda_2(t) \ldots$

Ground state : $|\phi(t)\rangle$

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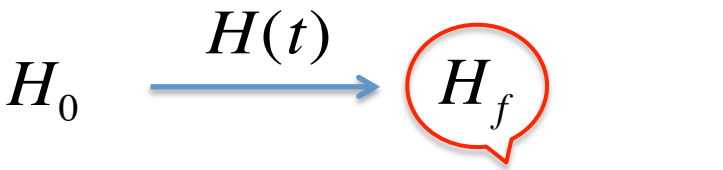


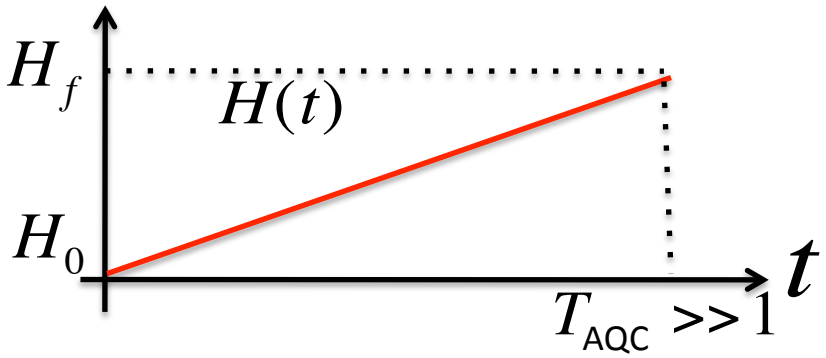


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Ground state : $|\phi(t)\rangle$

**Adiabatic theorem** (Born-Fock)

A quantum system initially in its ground state will remain in its instantaneous ground state in the limit in which the perturbation rate vanishes ($T >> 1$).

# Quantum Adiabatic Approximation

We rescale the time so that $s = t/T_{\mathrm{AQC}}$, $0 \le s \le 1$

$$H_0 = H(0) \longrightarrow H(1) = H_f$$

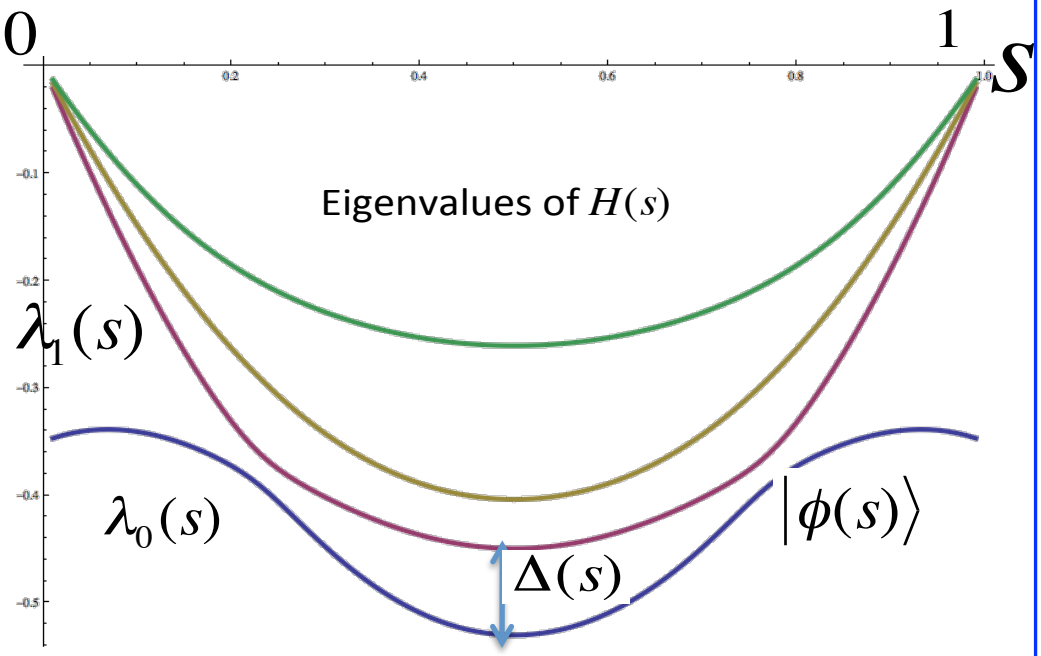
$$H(s(t)) = H(t)$$

$$|\phi(s(t))\rangle = |\phi(t)\rangle$$

**Standard quantum adiabatic approximation [1]**

$$T_{\mathrm{AQC}} \propto \max_s \left\{ \frac{\|\partial_s^2 H(s)\|}{(\Delta(s))^2}, \frac{\left(\|\partial_s H(s)\|\right)^2}{(\Delta(s))^3} \right\}$$

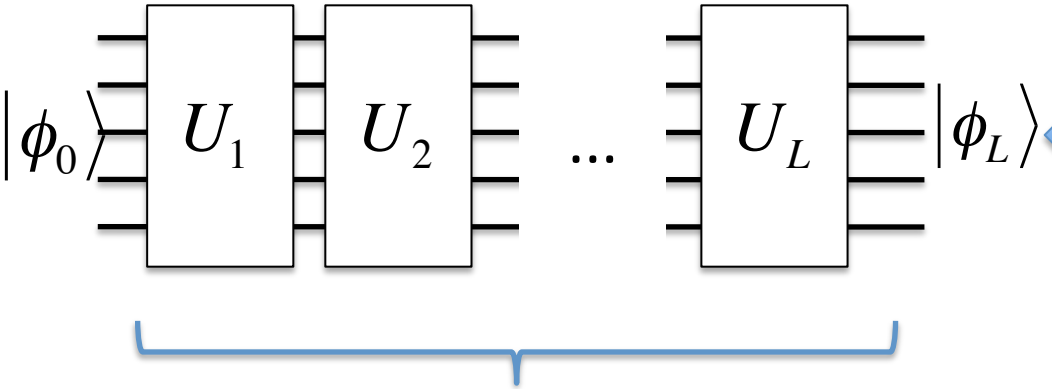
$$|\psi(T_{\mathrm{AQC}})\rangle \approx_\varepsilon |\phi(1)\rangle$$



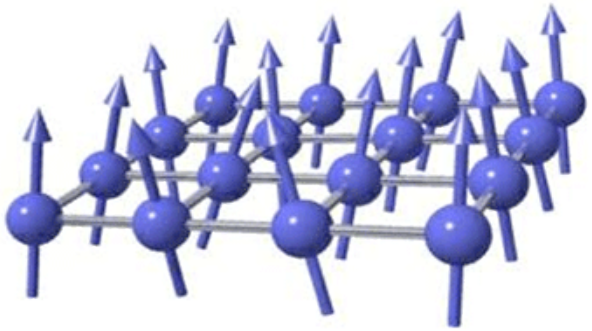

[1] S. Jansen, M. Ruskai, and R. Seiler, J. Math. Phys.48, 102111(2007). +++

# Adiabatic Quantum Computing: Equivalence and complexity

Any quantum circuit can be simulated with an adiabatic evolution [2]:




Simple one or two-qubit gates



$$H(1) = f(U_1,\ldots,U_L)\ ;\ \left|\langle\phi_L|\psi(1)\rangle\right| \approx 1$$

Size of the system is polynomial in $L$

[2] D. Aharonov, et.al., SICOMP 37, 166 (2007). A. Mizel, D. Lidar, M. Mitchell, PRL 99, 070502 (2007).

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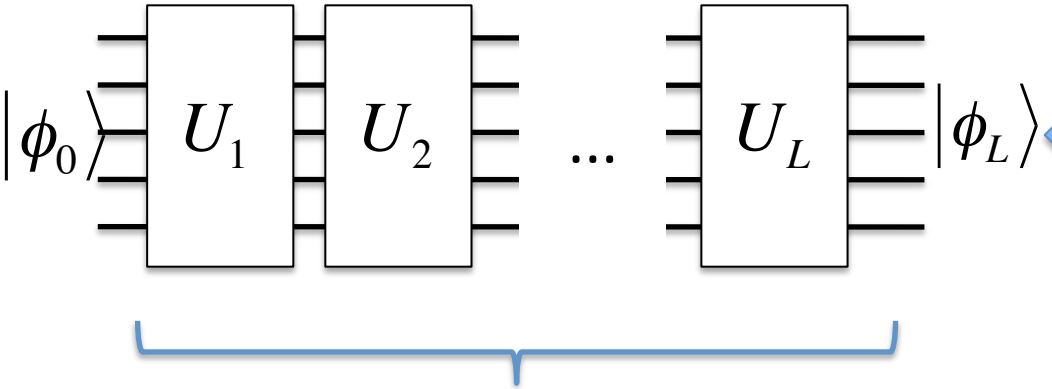


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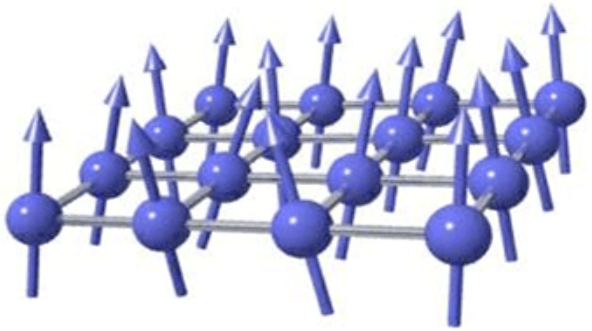

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How many elementary gates do we need to simulate the evolution operator?

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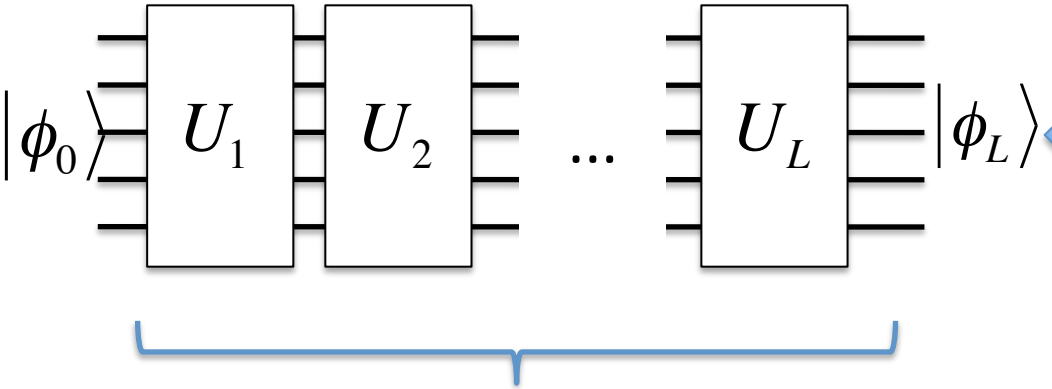


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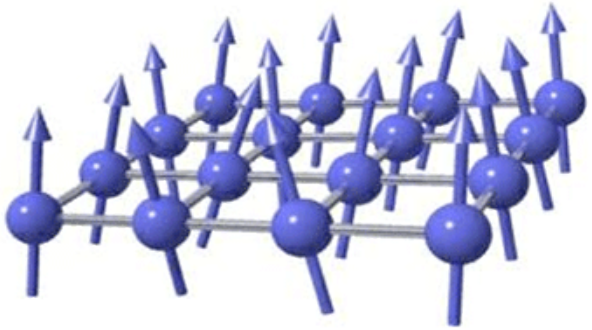

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How many elementary gates do we need to simulate the evolution operator?

- Sparse
- Bounded norm

$$L = \text{poly}(T)$$

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# Adiabatic Quantum Computing: Why?

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**Why do we really care about AQC?**

Inherent robustness? Fault tolerance?

Natural model for solving certain problems, such as combinatorial optimization

**Provable quantum speedups?**

**Faster ways to prepare the final ground state?**

evolution time $T << T_{\text{AQC}}$ ???

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Issues with standard adiabatic approximations:

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# Quantum Adiabatic Approximation: New methods

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In the last few years, we developed several methods to prepare the final ground state in time much less than that given by AQC. Our methods resulted in provable quantum speedups of several classical algorithms, such as Monte Carlo.

- Method 1: Evolution randomization (Boixo, Knill, Xu)
- Method 2: Measurement based (Boixo, Knill)
- Method 3: Diabatic transitions (Nagaj, Kieferova)

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- Method 1: Evolution randomization (Boixo, Knill, Xu)
- Method 2: Measurement based (Boixo, Knill)

"adiabatic state transformations"

- Method 3: Diabatic transitions (Nagaj, Kieferova)

"quantum annealing"

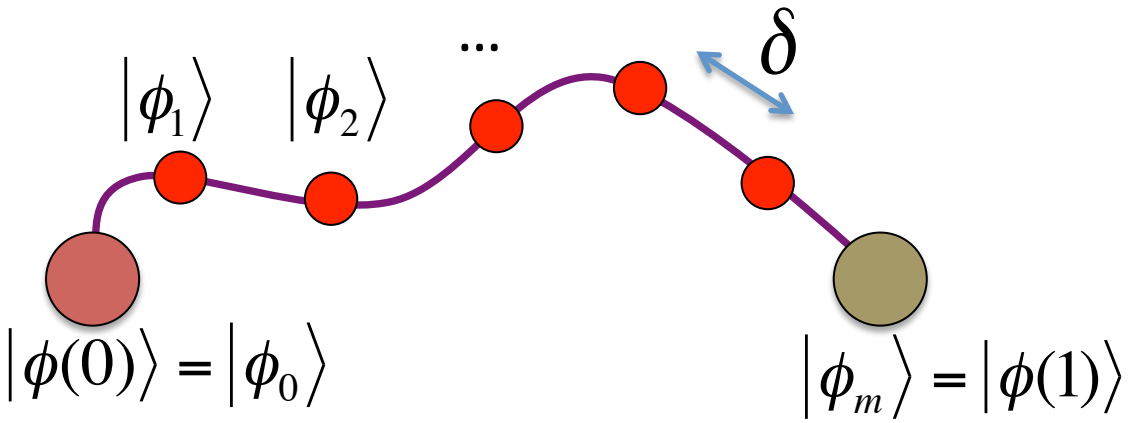
# 1. Evolution randomization [3]

As in AQC, the goal is to prepare the final ground state (or any eigenstate) from the initial one by sequentially preparing the ground states (eigenstates) along the path.

Assume a discretization for the eigenpath (path of ground states)

$$\{0 = s_0, s_1, \ldots, s_m = 1\}$$

$$|\phi_j\rangle = |\phi(s_j)\rangle$$




[3] S Boixo, E Knill, **RDS**, QIC 9, 0833 (2009). Chiang, Xu, **RDS**, PRA 89, 012314 (2014)

# 1. Evolution randomization: “Measurements”

Discretization for the eigenpath $\{0 = s_0, s_1, \ldots, s_m = 1\}$

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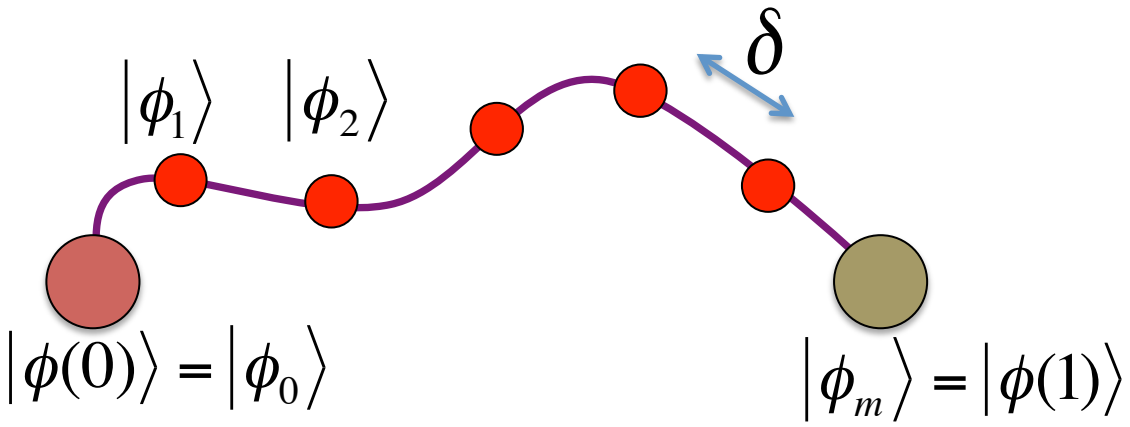
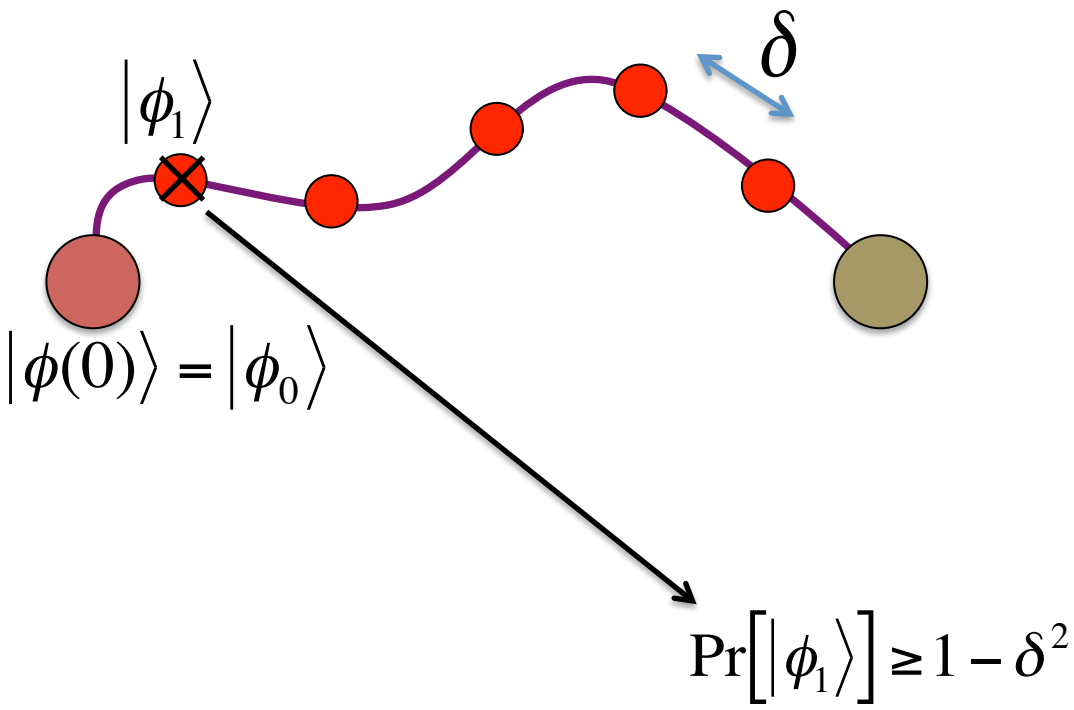



$$\left|\langle \phi_{j-1} | \phi_j \rangle\right| \geq 1 - \delta^2$$

A sequence of projective measurements into the ground states prepares the final ground state with high probability

# 1. Evolution randomization: “Measurements”

$$\left|\phi_j\right\rangle = \left|\phi(s_j)\right\rangle$$



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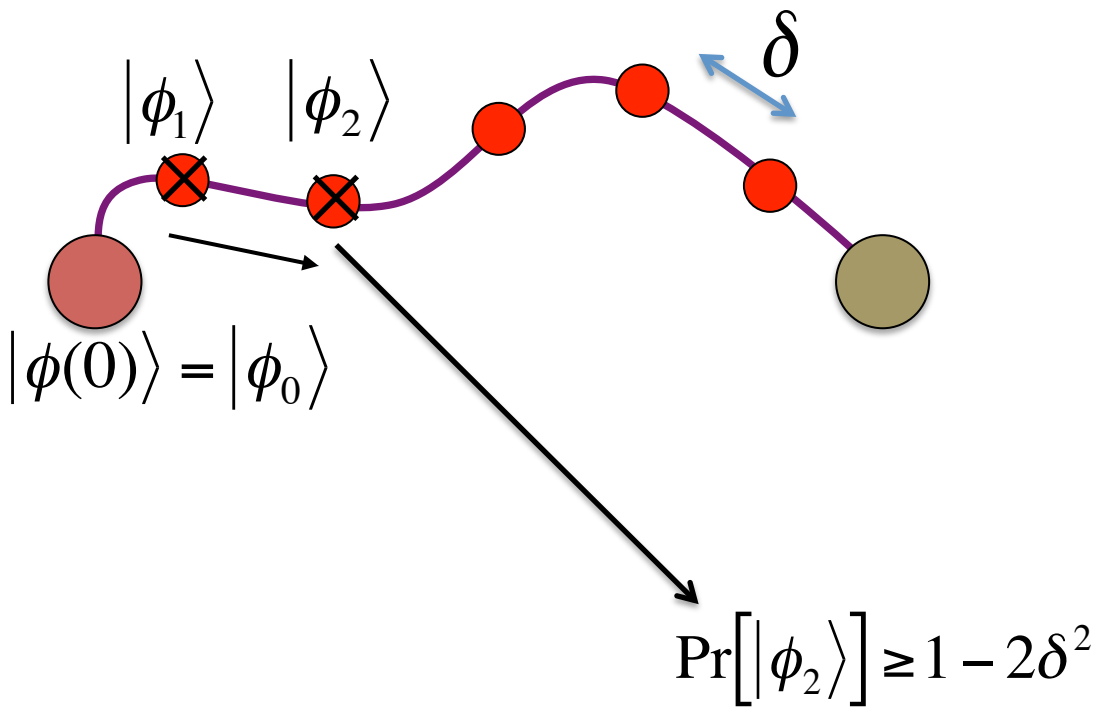
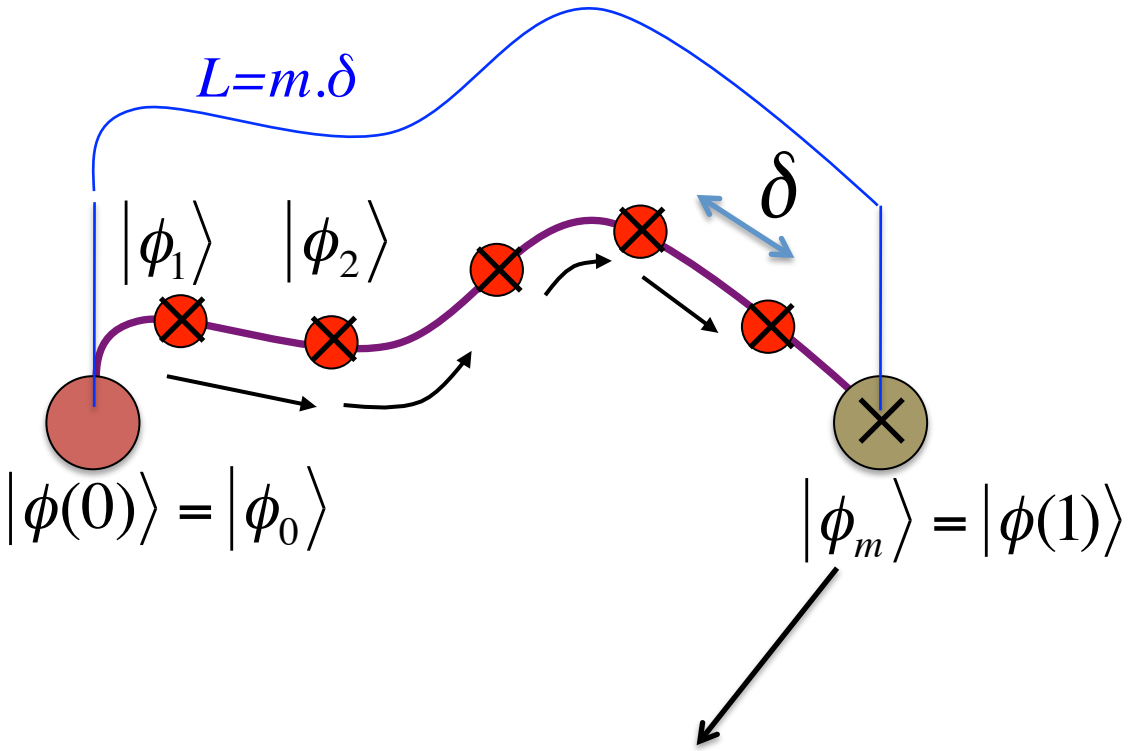


# 1. Evolution randomization: Path length

$$|\phi_j\rangle = |\phi(s_j)\rangle$$



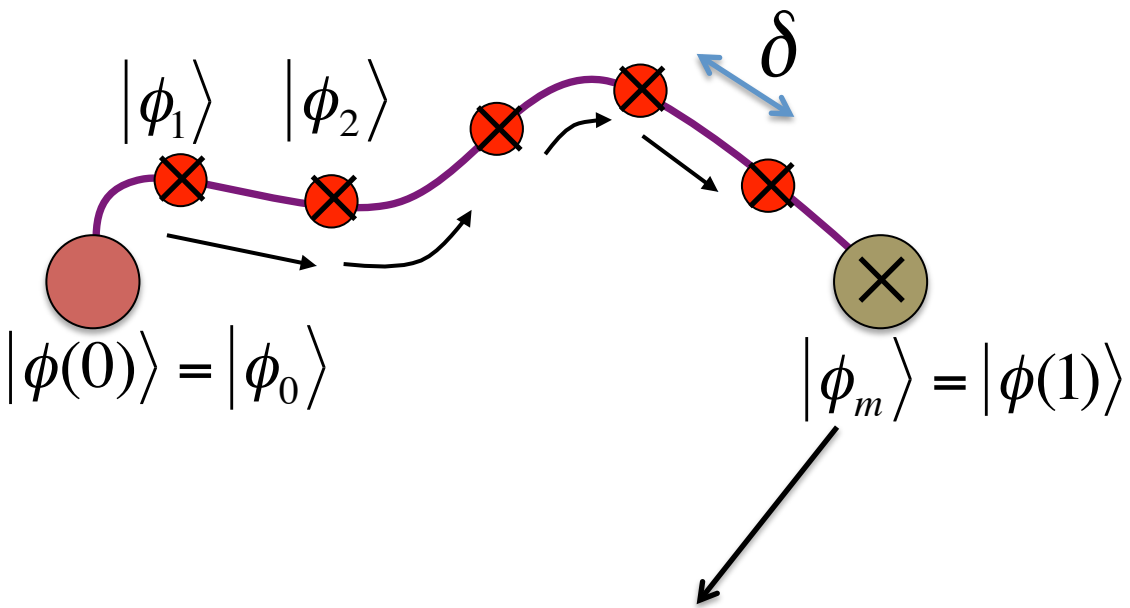

$$\Pr\left[|\phi_m\rangle\right] \geq 1 - m\delta^2 = 1 - L\delta$$

Path length

➡ It suffices to pick $\delta = \varepsilon/L$ for overall error $\varepsilon$

# 1. Evolution randomization: Cost

$$|\phi_j\rangle = |\phi(s_j)\rangle$$




$$\Pr\left[|\phi_m\rangle\right] \geq 1 - m\delta^2 = 1 - \underbrace{L\delta}_{\varepsilon}$$

It suffices to pick $\delta = \varepsilon/L$ for overall error $\varepsilon$

Number of projective measurements:

$$m = \frac{L^2}{\varepsilon} \; ; \; m = \frac{\int_0^1 ds \left|\langle \partial_s \phi(s) | \partial_s \phi(s) \rangle\right|}{\varepsilon}$$

# 1: Evolution randomization: Simulation of measurements

Simulation of projective measurement by evolution randomization

$$e^{-iH_j t}\left|\phi_j\right\rangle\left\langle\phi_j\right|e^{+iH_j t} = \left|\phi_j\right\rangle\left\langle\phi_j\right|$$

$$e^{-iH_j t}\left|\phi_j\right\rangle\left\langle\phi_j^{\perp}\right|e^{+iH_j t} = e^{i\Delta' t}\left|\phi_j\right\rangle\left\langle\phi_j^{\perp}\right|$$

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$$\Delta' \geq \Delta$$

Orthogonal eigenstate

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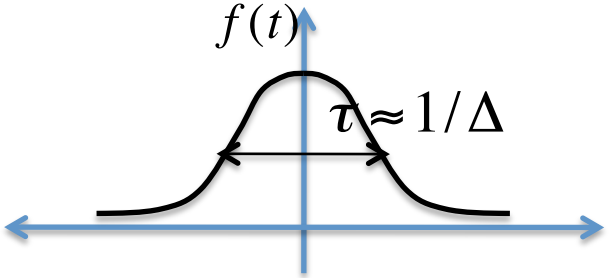


$\exists\ f(t)$ such that

$$\int_0^{\tau} dt.f(t)e^{-iH_j t}\left|\phi_j\right\rangle\left\langle\phi_j^{\perp}\right|e^{+iH_j t} = 0$$

Evolution randomization can eliminate coherences, thus simulating a projective measurement

$\tau \approx 1/\Delta$ suffices from Fourier analysis

global cost : $T_{\text{rand}} = \tau m \propto \dfrac{L^2}{\varepsilon\Delta}$

# 1. Evolution randomization: Total cost

Simulation of projective measurement by evolution randomization

$$e^{-iH_j t}\left|\phi_j\right\rangle\left\langle\phi_j\right|e^{+iH_j t} = \left|\phi_j\right\rangle\left\langle\phi_j\right|$$

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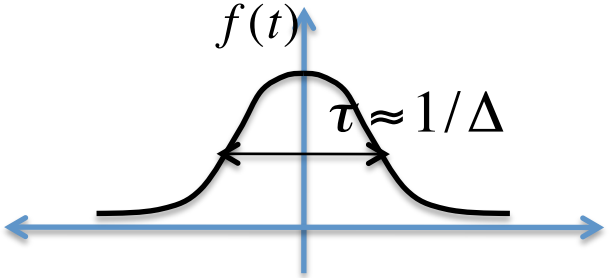
Orthogonal eigenstate

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global cost : $T_{\text{rand}} = \tau m \propto \dfrac{L^2}{\varepsilon\Delta}$

$T_{\text{rand}} << T_{\text{AQC}}$ in many examples

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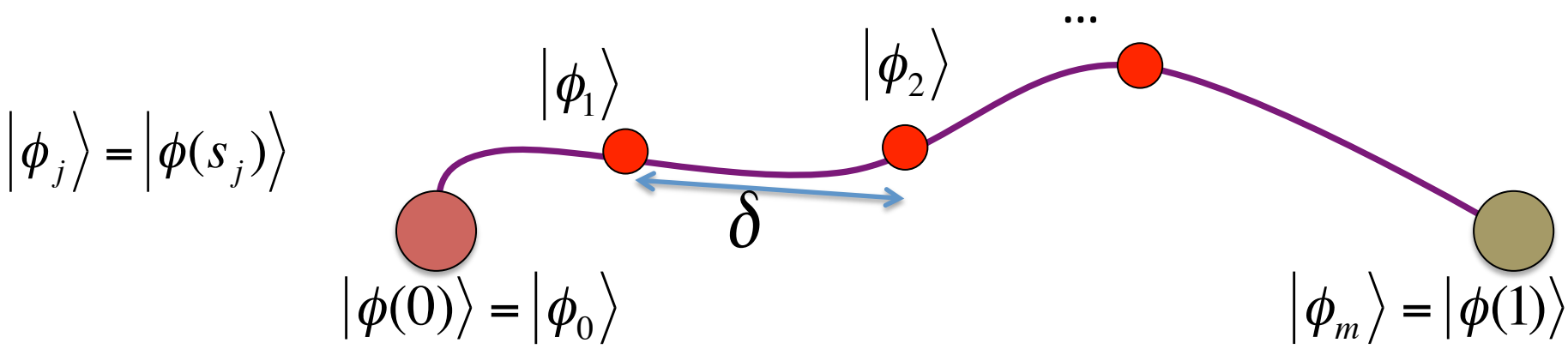
# 2. Measurement based

The main different with the other method is that we will simulate the projective measurements in the ground states in a different way. Rather than needing $\delta=\varepsilon/L$, we will be able to choose a constant $\delta$ and reduce the cost (number of points in the discretization).



# 2. Measurement based: Basic steps

**One-step state transformations**

Goal : prepare $\left|\phi_j\right\rangle$ from $\left|\phi_{j-1}\right\rangle$ using reflection

oracles $R_{j-1} = 1 - 2\left|\phi_{j-1}\right\rangle\left\langle\phi_{j-1}\right|$ and $R_j = 1 - 2\left|\phi_j\right\rangle\left\langle\phi_j\right|$

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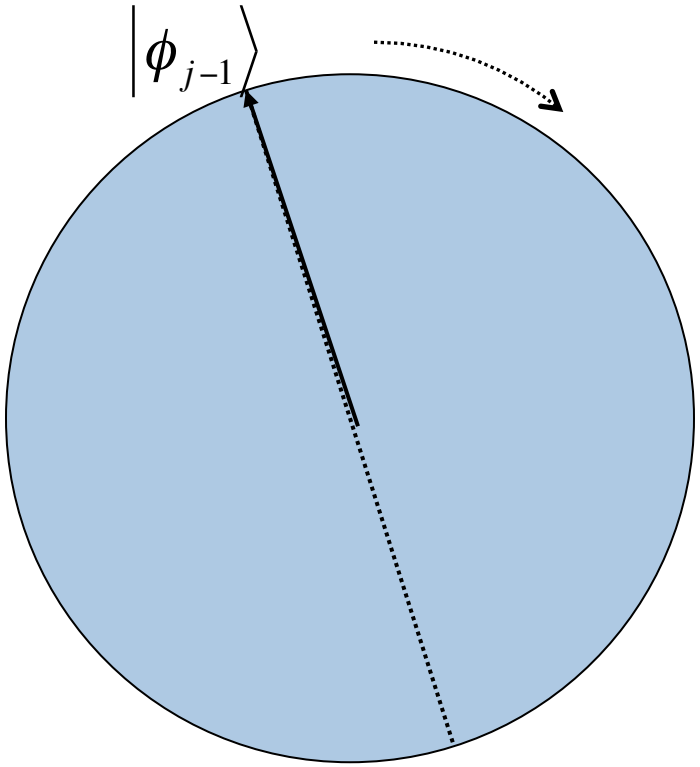


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Pseudocode:

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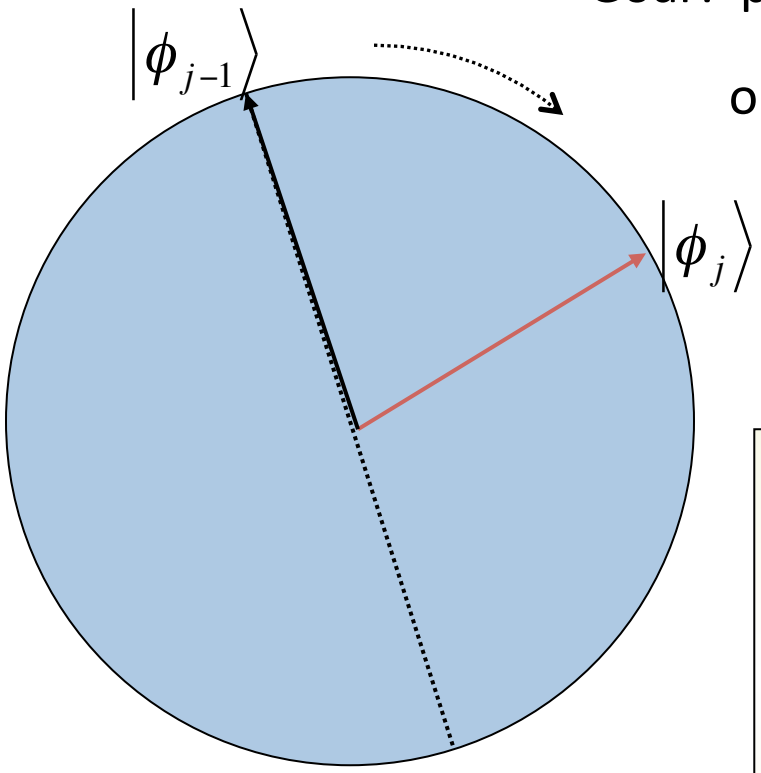


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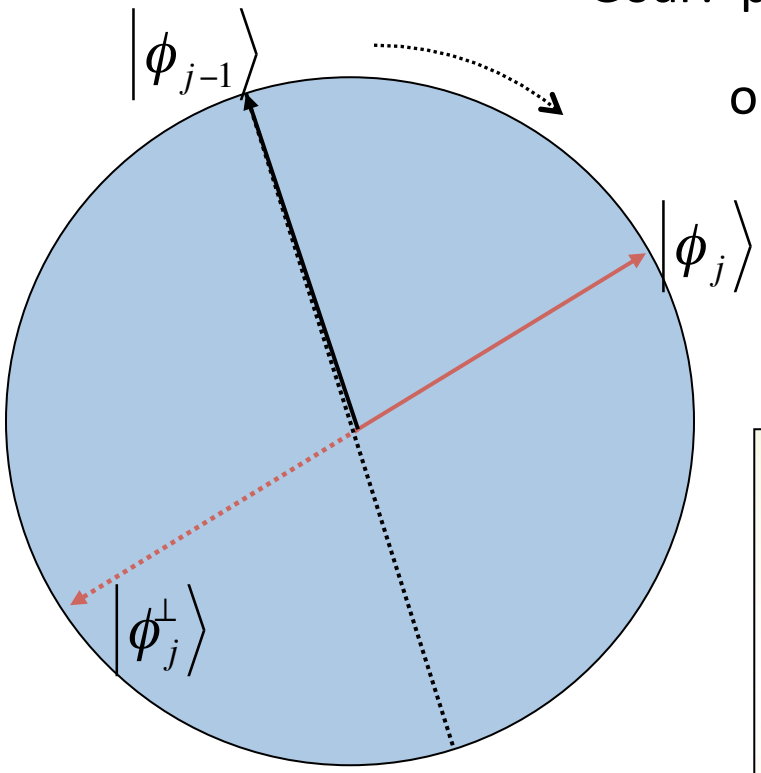


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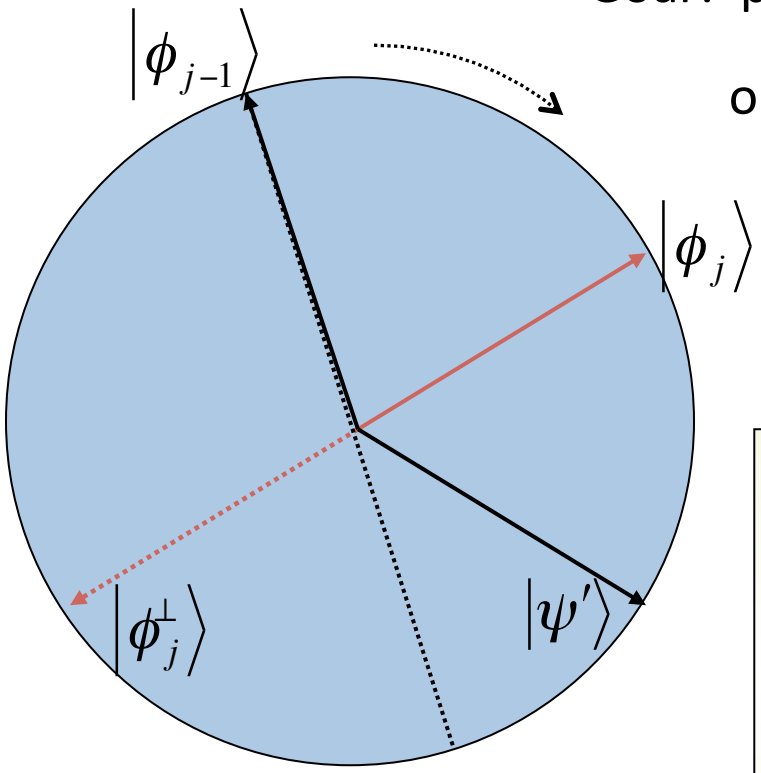


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Pseudocode:

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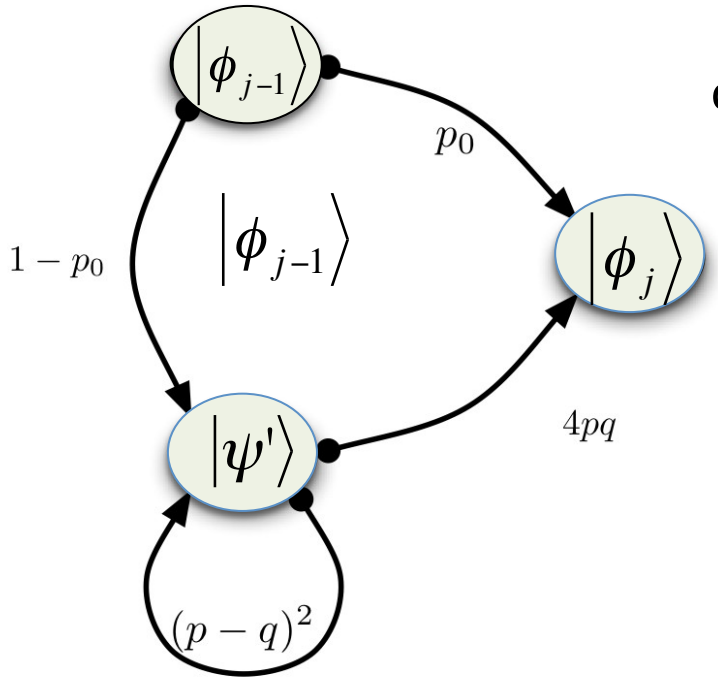
2. If successful : STOP
3. Else : Apply $R_{j-1}$
4. Go to 1.

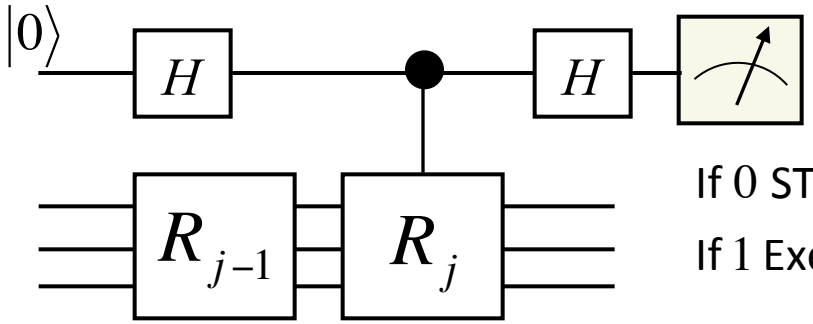
# 2. Measurement based: Basic steps

**One-step state transformations**

Goal : prepare $|\phi_j\rangle$ from $|\phi_{j-1}\rangle$ using reflection

oracles $R_{j-1} = 1 - 2|\phi_{j-1}\rangle\langle\phi_{j-1}|$ and $R_j = 1 - 2|\phi_j\rangle\langle\phi_j|$




If 0 STOP

If 1 Execute again

$$p_0 = \left|\langle\phi_j|\phi_{j-1}\rangle\right|^2 \geq 1-\delta^2 \;\; ; \;\; p = \left|\langle\psi'|\phi_j\rangle\right|^2 \;\; ; \;\; q = 1-p$$

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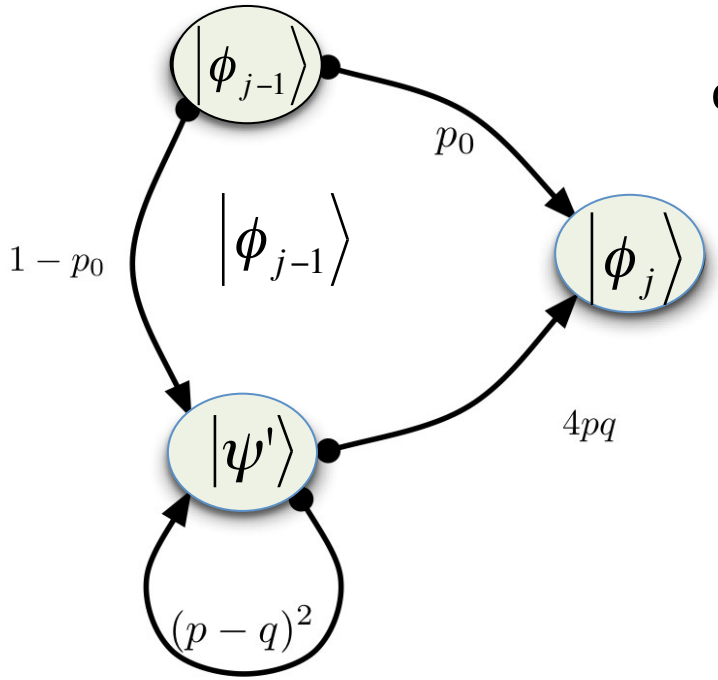


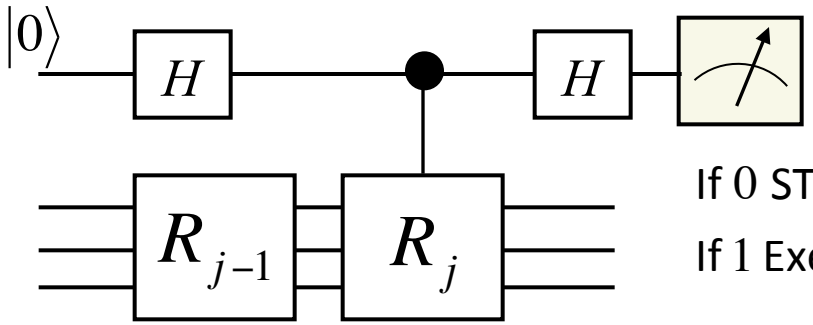


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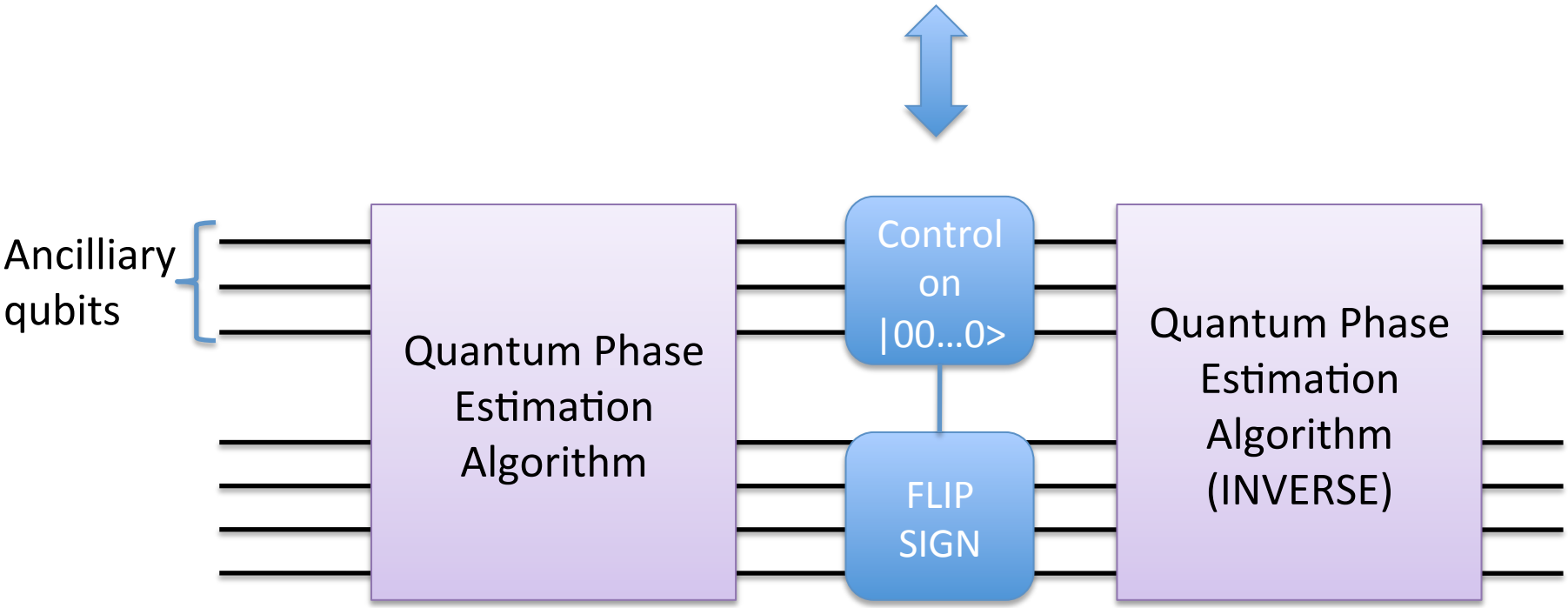
If $p_0 > 1/3 \Rightarrow \langle n\rangle$ is order 1 and all moments are bounded

# 2. Measurement based: How to implement reflections

Assume we know the ground state energy $H_j \left|\phi_j\right\rangle = 0$

$$R_j = 1 - 2\left|\phi_j\right\rangle\left\langle\phi_j\right|$$




The phase estimation algorithm needs to resolve eigenvalues above the gap. Thus, it requires a cost of 1/Δ:

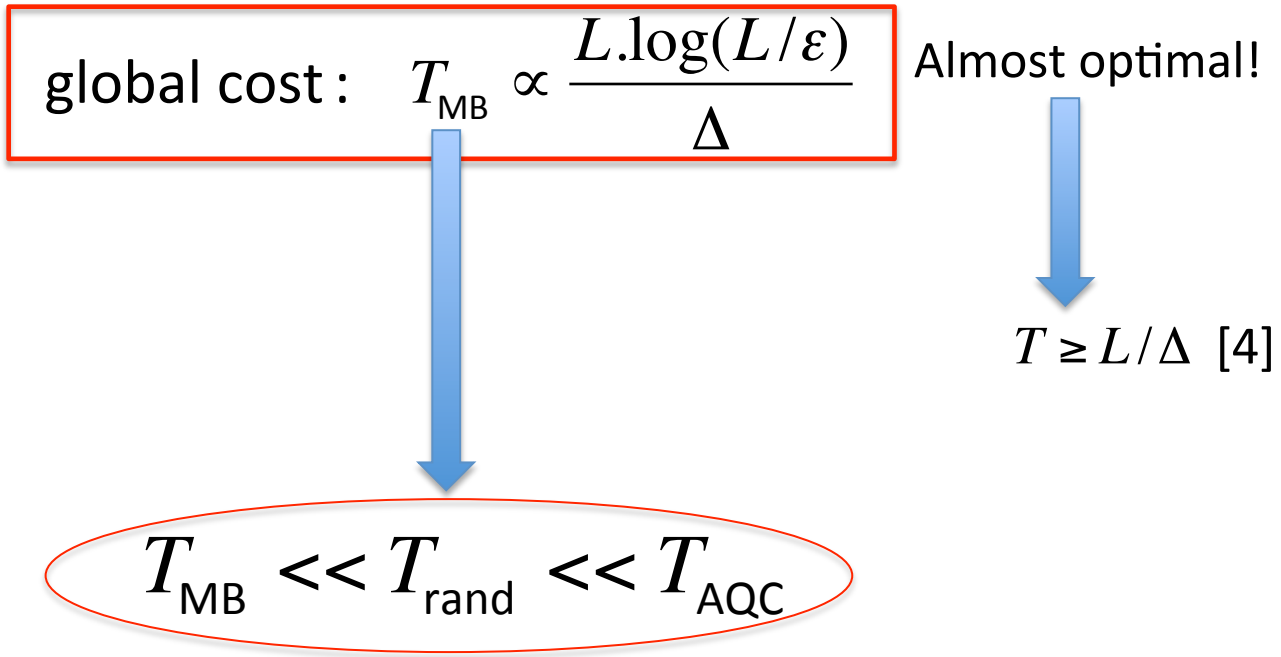
$$e^{-iH_j t} \;;\; t \approx 1/\Delta$$

# 2. Measurement based: Cost

global cost : $T_{\mathrm{MB}} \propto \dfrac{L.\log(L/\varepsilon)}{\Delta}$

Almost optimal!

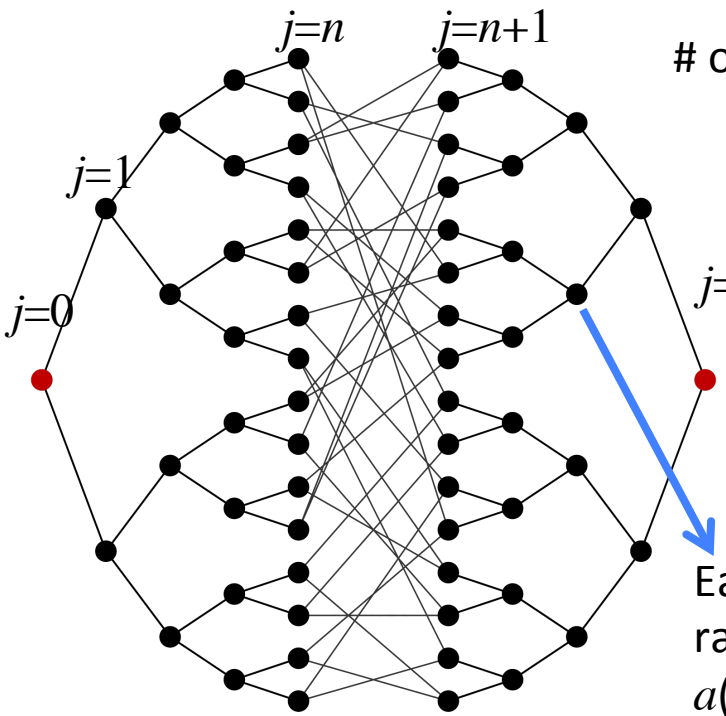
# 2. Measurement based: Cost




[4] S. Boixo and R. Somma, PRA 81, 032308 (2010).

# 3. Diabatic transitions: Glued-Trees problem

*The (randomly) Glued-Trees Problem (GT) [5]*




# of vertices: $2^{n+2} - 2$

Each vertex is randomly labeled as $a$(VERTEX)

(given oracle access to $A$)

[5] A. M. Childs, R. Cleve, et.al., in Proc. 35th Annual ACM Symp. Theo. of Comp., p. 59 (2003)

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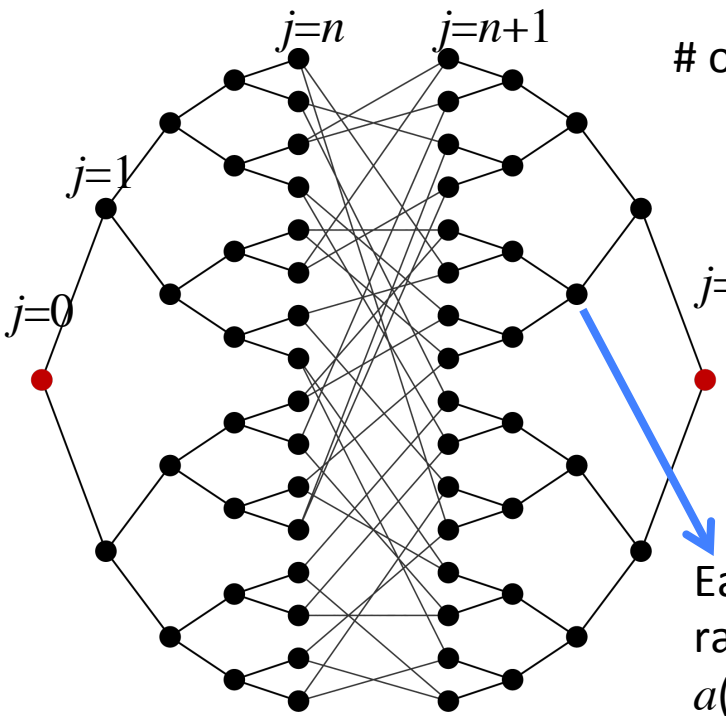


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Classically it takes time exponential in $n$ to find the exit. Quantumly it can be done in time polynomial in $n$

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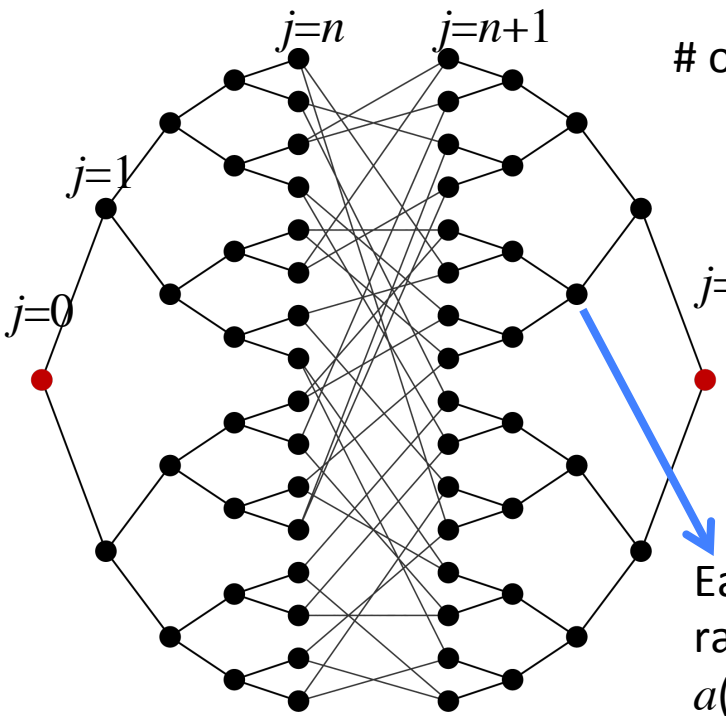


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Can we solve this problem adiabatically?

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# 3. Diabatic transitions: Glued-Trees problem

Exponential speedups [6]

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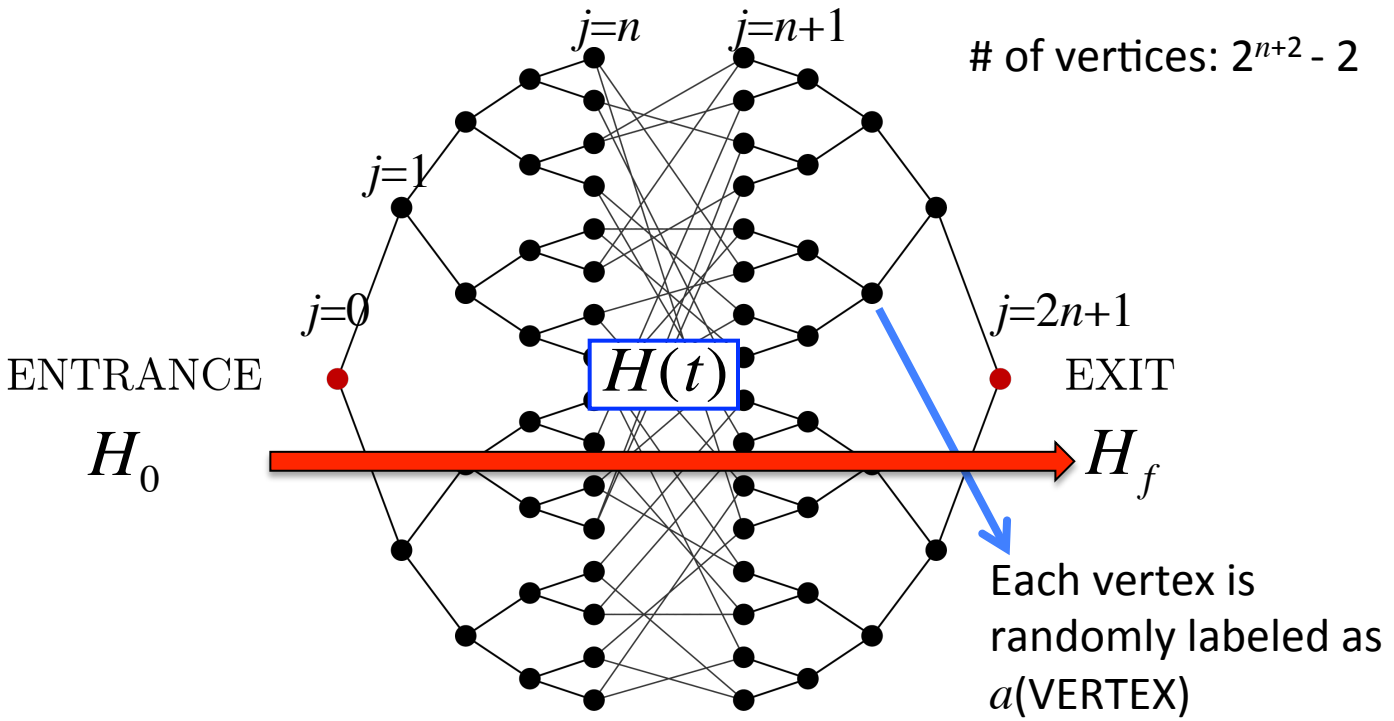


$$H(s) = (1-s)|\text{ENTRANCE}\rangle\langle\text{ENTRANCE}| + s(1-s)A + s|\text{EXIT}\rangle\langle\text{EXIT}|$$

[6] "Quantum speedup by quantum annealing", RS, Nagaj, Kieferova, PRL 109, 050501 (2012)

# 3. Diabatic transitions: Quantum annealing

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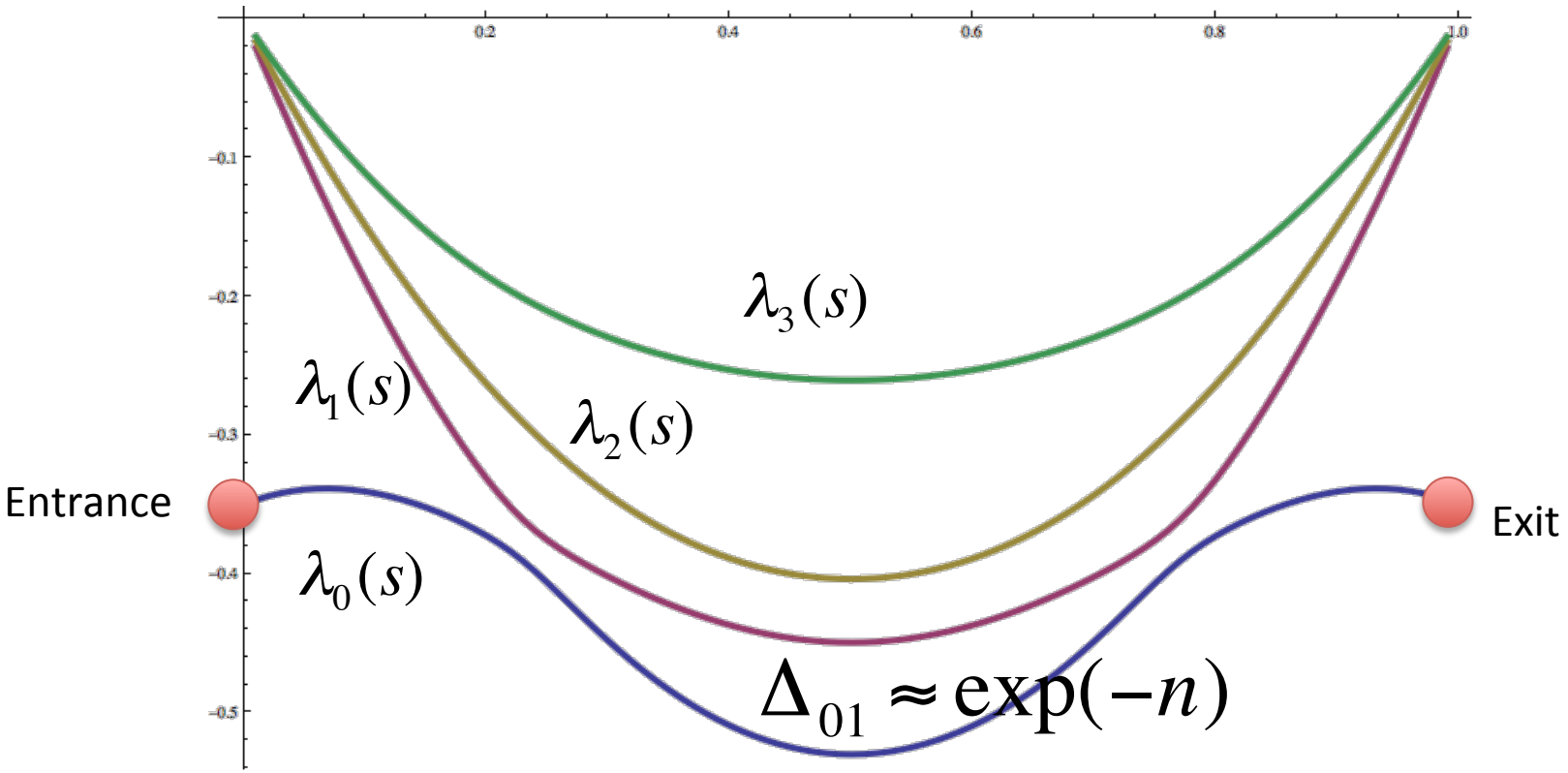

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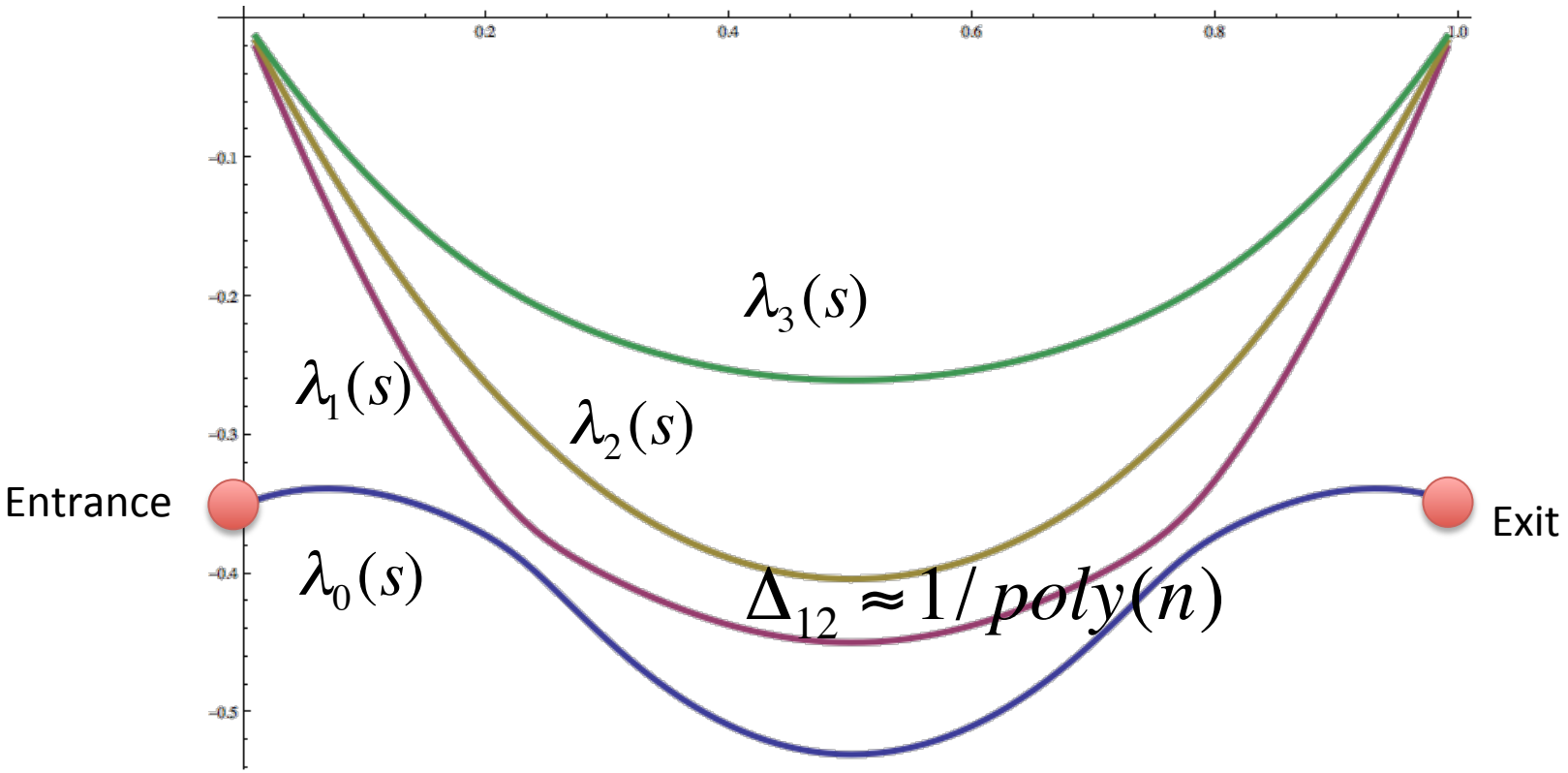


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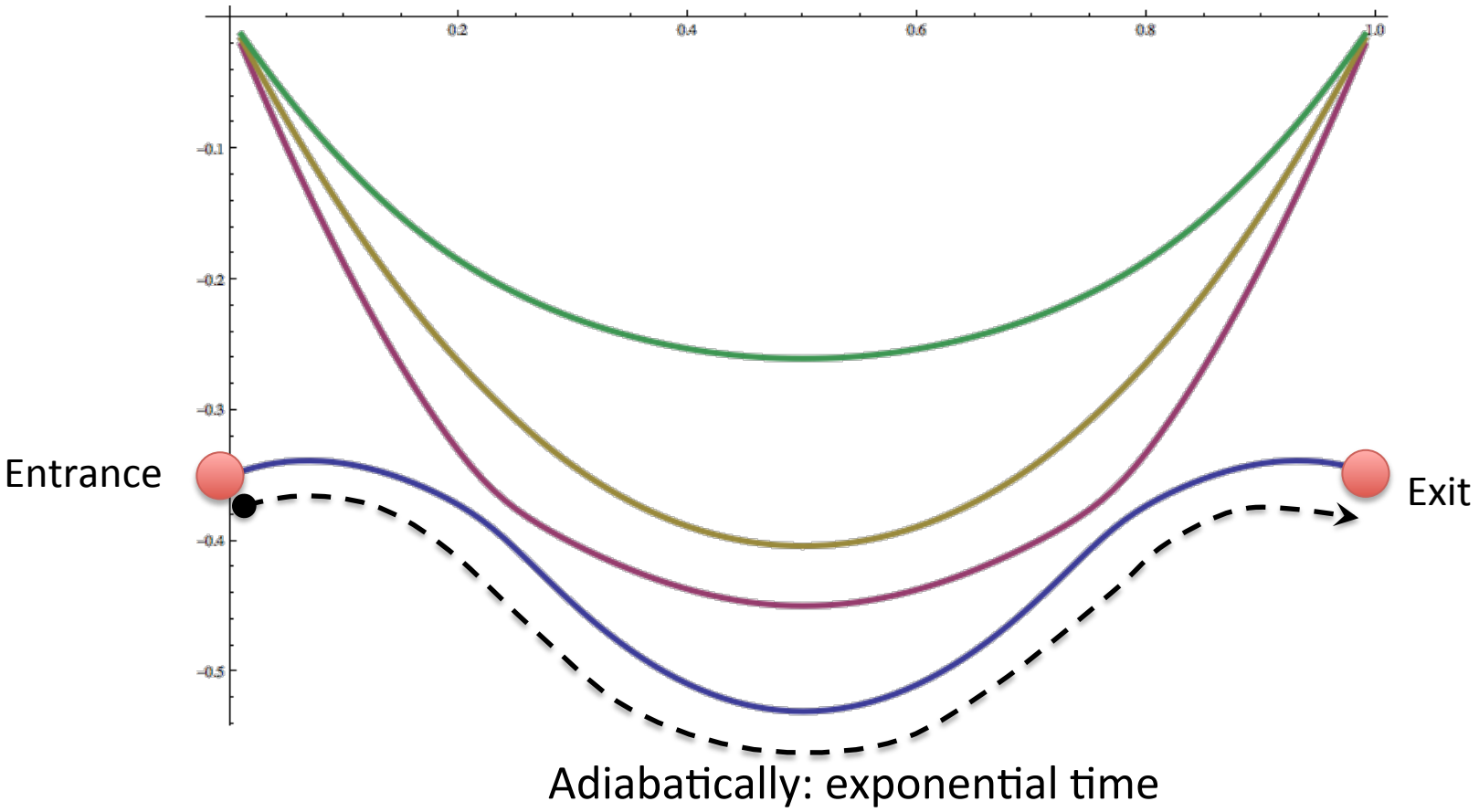


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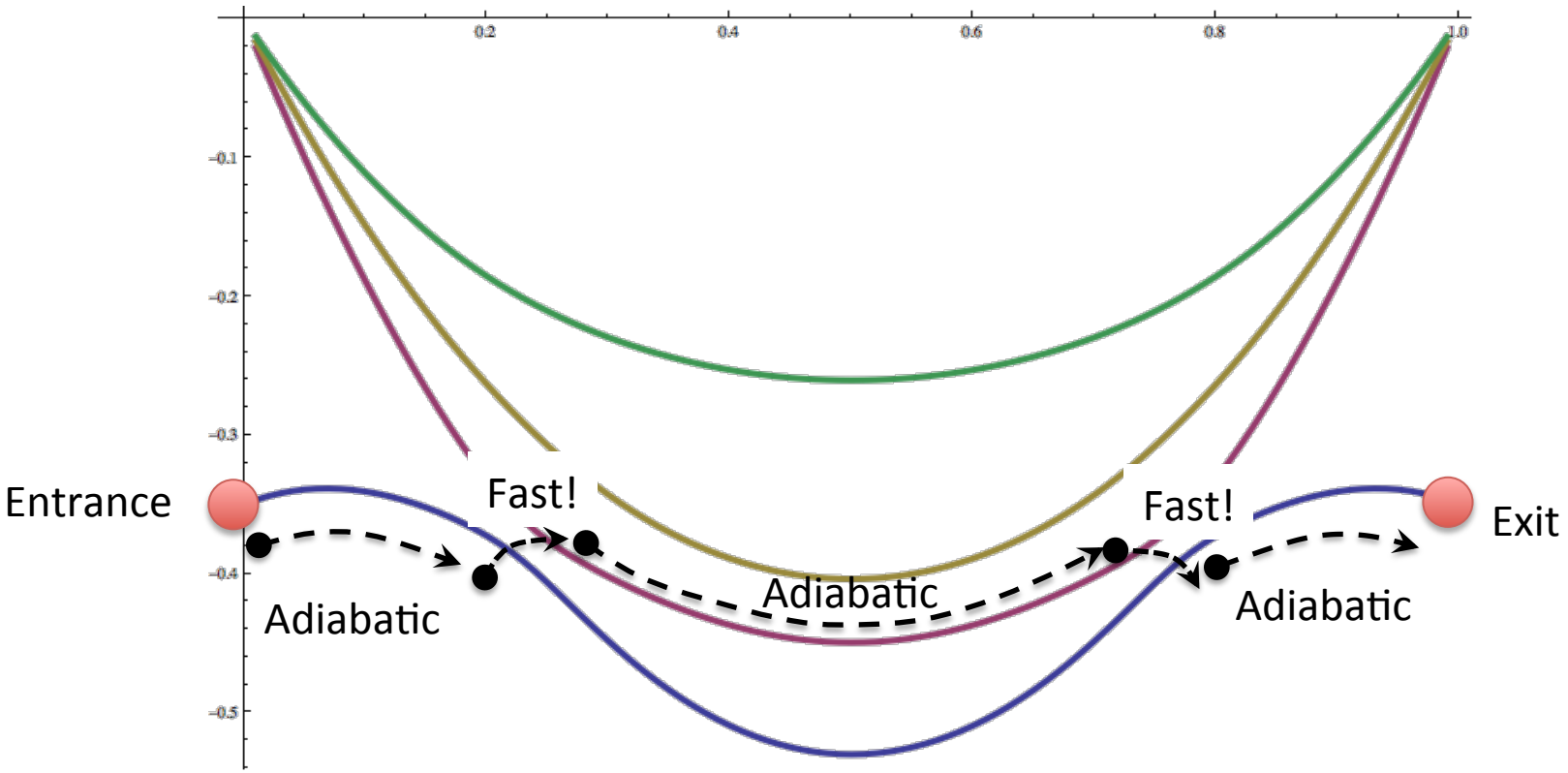


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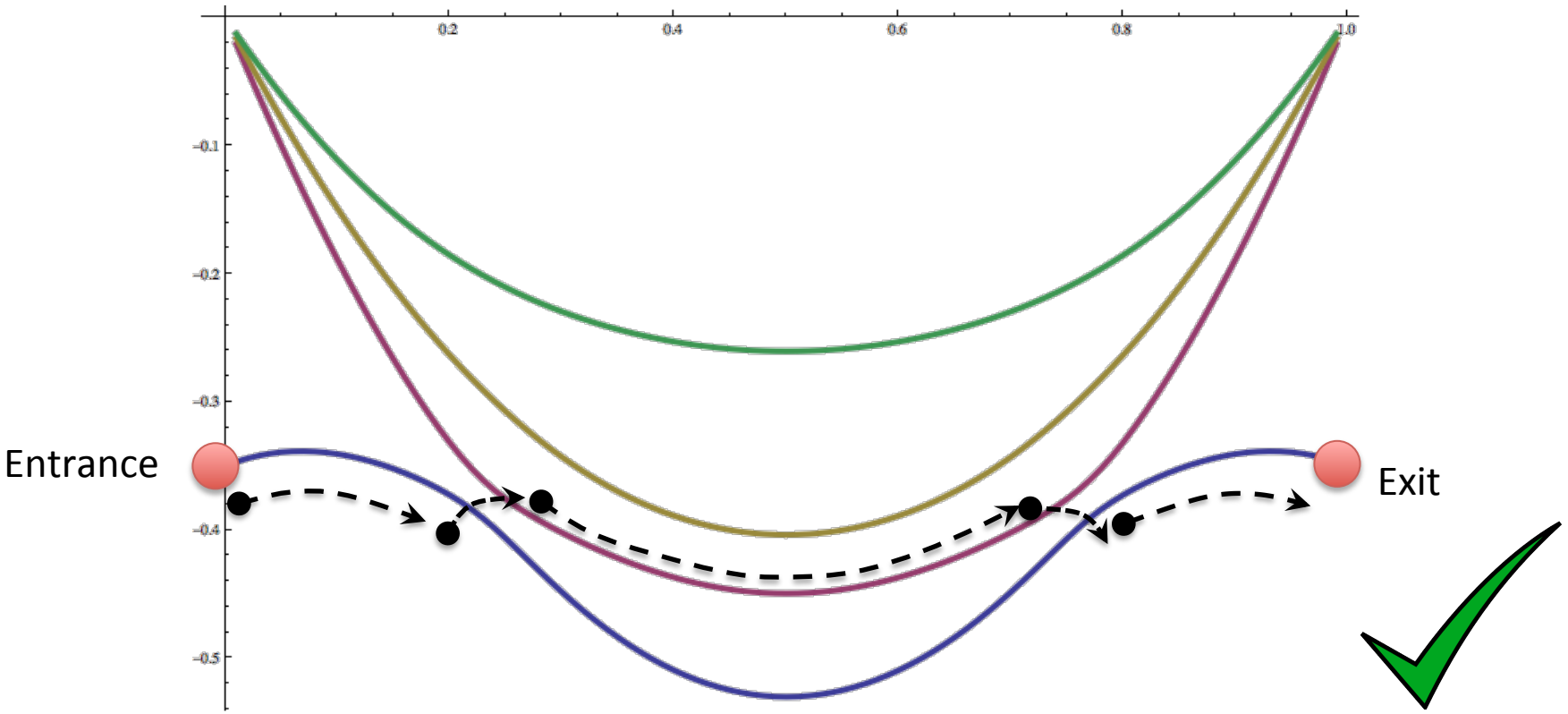


**Diabatic transitions: polynomial time !!**

**Provable quantum speedup**

[6] "Quantum speedup by quantum annealing", RS, Nagaj, Kieferova, PRL 109, 050501 (2012)

# 3. Diabatic transitions: Importance & generalizations

Why is the result important? (Additional reasons)

• Recently, motivated by our results, a similar property on the spectrum of other Hamiltonians for solving MAX 2 SAT was shown [7]. This provided faster annealing algorithms for this problem.

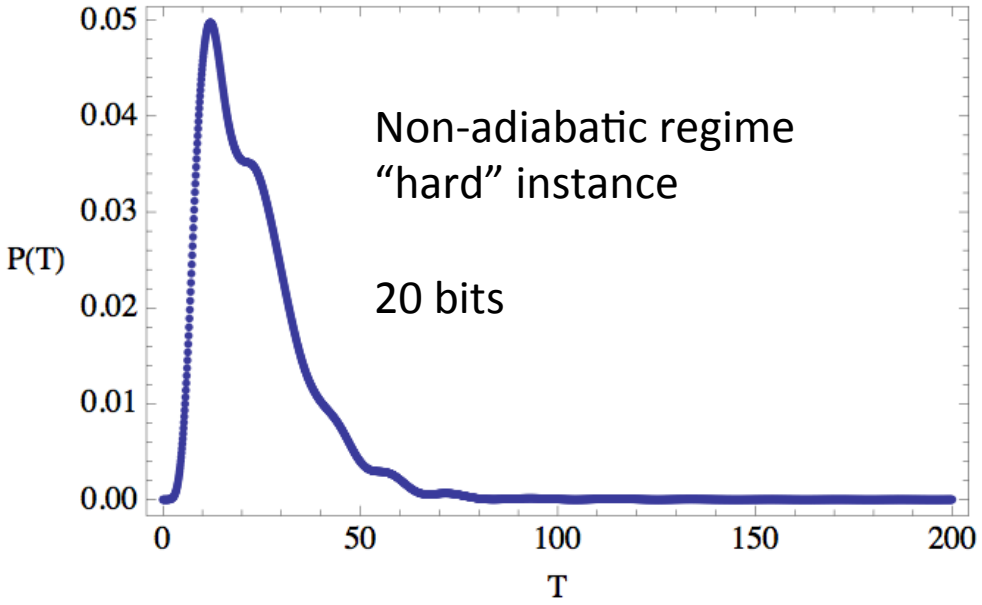


FIG. 2: The success probability as a function of total evolution time $T$ for instance #1.

[7] E.Crosson, E. Farhi, C. Lin, H. Lin, P. Shor, arXiv: 1401.7320 (2014)

# 3. Diabatic transitions: Importance & generalizations

Why is the result important? (Additional reasons)

• The Hamiltonians involved do not suffer from the so-called sign problem. Then, classical techniques such as quantum Monte Carlo can be used in these cases. Can quantum annealing outperform quantum Monte Carlo? When the gaps are big, this question remains unanswered.

# Conclusions

• We presented methods for adiabatic state transformations that, in some cases, are (almost) optimal and achieve a cost of $L/\Delta$. Can we always achieve such cost?

• We presented a method to avoid the overheads due to extremely small gaps in adiabatic state transformations. This method aims at adiabatically decoupling subspaces and uses diabatic transitions to excited states.

• The techniques we presented allow for **provable** polynomial and exponential speedups.